The IFF Type Namespace

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0.1 Type Sets

Basics. There is an IFF set set of all type *set*s. Any type set is itself an IFF set; that is, the IFF set of all type sets is an (implicit) subset of the set of all IFF sets.

```
(iff:set set)
(forall ((set ?X)) (iff:set ?X))
```

There is a binary *isomorphic* endorelation between pairs of type sets, that are linked by an bijection[1]. Since there is no notion of binary relation specified at the IFF level, at the type level we use the extent set of the isomorphism relation[2] to indirectly specify it. The isomorphic endorelation is an equivalence relation (reflexive, symmetric and transitive), since identities are bijections, bijections have an inverse and bijections are closed under composition.

```
(iff:set isomorphic-relation)
(forall ((isomorphic-relation ?XY)) (type.dgm.pr.obj:set-pair ?XY))
(forall ((set ?X) (set ?Y))
    (<=> (isomorphic-relation [?X ?Y])
         (exists ((type.ftn:bijection ?f))
             (and (= ?X (type.ftn:source ?f)) (= ?Y (type.ftn:target ?f))))))
(forall ((set ?X))
    (isomorphic-relation [?X ?X]))
(forall ((set ?X) (set ?Y))
    (=> (isomorphic-relation [?X ?Y])
        (isomorphic-relation [?Y ?X])))
(forall ((set ?X) (set ?Y) (set ?Z))
    (=> (and (isomorphic-relation [?X ?Y]) (isomorphic-relation [?Y ?Z]))
        (isomorphic-relation [?X ?Z])))
```

The following is taken from the book *Sets for Mathematics* (2003) by Lawvere and Rosebrugh.

Fact 1 (Galileo) *The set* $\mathsf{Natno} = \aleph = \{0, 1, 2, \cdots\}$ *of all natural numbers is isomorphic to the set* $\mathsf{Sqr} = \{0, 2, 4, \cdots\} \subset \mathsf{Natno}$ *of all square whole numbers (a proper subset).*

Proof. Use the squaring function $(\text{-}) : \mathsf{Natno} \xrightarrow{\sim} \mathsf{Sqr}$ and its inverse the square root function $\surd : \mathsf{Sqr} \xrightarrow{\sim} \mathsf{Natno}$. ∎

This observation by Galileo was generalized into a definition by Dedekind.

Definition 1 (Dedekind) *A set X is* finite *when all injections on X are bijections. A set X is* infinite *when there is at least one injection on X that is not surjective.*

---

[1] Here we follow and quote from the discussion on the topic of isomorphism and Dedekind finiteness as presented in Lawvere and Rosebrugh [1]. "The notation $f : X \xrightarrow{\sim} Y$ means that f is an isomorphism. One type set X is *isomorphic* to a type set Y when there is at least one isomorphism (type bijection) from X to Y. This definition of isomorphism is used in all categories, but in a category of abstract sets and arbitrary functions the two type sets X and Y are said to be *equinumerous* or to *have the same cardinality*." As Lawvere and Rosebrugh point out, the isomorphism of abstract sets offers a method to study equinumerosity without counting, a fact systematically used by Cantor.

[2] We use this idea of "extent serving as proxy" for other relations and orders, such as subset, delimitation, (optimal-)restriction, abridgment, function (element) belonging, predicate (part) inclusion and the member relations between functions and predicates and between spans and relations. This is a small part of the bootstrapping mechanism of the IFF metashell.

These are predicates (adjectives) that refer to (modify) sets (nouns), the genus, and result in subsets (noun phrases), the differentia.

```
(iff:set finite-set)
(forall ((finite-set ?X)) (set ?X))
(forall ((set ?X))
    (<=> (finite-set ?X)
         (forall ((type.ftn:injection ?f)
                 (= (type.ftn:source ?f) ?X)
                 (= (type.ftn:target ?f) ?X))
             (type.ftn:bijection ?f))))

(iff:set infinite-set)
(forall ((infinite-set ?X)) (set ?X))
(forall ((set ?X))
    (<=> (infinite-set ?X)
         (not (finite-set ?X))))
```

A type set X is relatively *small* when it is a meta set. That is, the predicate "small" has as its genus the set of all type sets and as its differentia the set of all meta sets.

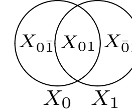
```
(iff:set small-set)
(forall ((small-set ?X)) (set ?X))
(forall ((set ?X))
   (<=> (small-set ?X)
        (meta.set:set ?X)))
```

Any finite set is relatively small.

```
(forall ((finite-set ?X)) (small-set ?X))
```

For any pair of type sets X_0 and X_1, there are *binary union*, *binary intersection* and (binary) *difference* type sets defined as follows.

$$\begin{aligned} X_0 \cup X_1 &= \{y \mid y \in X_0 \text{ or } y \in X_1\} &= X_{0\bar{1}} \cup X_{01} \cup X_{\bar{0}1} \\ X_0 \cap X_1 &= \{y \mid y \in X_0 \text{ and } y \in X_1\} &= X_{01} \\ X_0 \setminus X_1 &= \{y \mid y \in X_0 \text{ and } y \notin X_1\} &= X_{0\bar{1}} \end{aligned}$$




```
(iff:function binary-union)
(= (iff:source binary-union) type.dgm.pr.obj:set-pair)
(= (iff:target binary-union) set)
(forall ((set ?X0) (set ?X1))
    (and (subset-relation [?X0 (binary-union [?X0 ?X1])])
         (subset-relation [?X1 (binary-union [?X0 ?X1])])
         (forall (((binary-union [?X0 ?X1]) ?x))
             (or (?X0 ?x) (?X1 ?x)))))

(iff:function binary-intersection)
(= (iff:source binary-intersection) type.dgm.pr.obj:set-pair)
(= (iff:target binary-intersection) set)
(forall ((set ?X0) (set ?X1))
    (and (subset-relation [(binary-intersection [?X0 ?X1]) ?X0])
         (subset-relation [(binary-intersection [?X0 ?X1]) ?X1])))
(forall ((set ?X0) (set ?X1) (?X0 ?x) (?X1 ?x))
    ((binary-intersection [?X0 ?X1]) ?x))

(iff:function difference)
```

```
(= (iff:source difference) type.dgm.pr.obj:set-pair)
(= (iff:target difference) set)
(forall ((set ?X0) (set ?X1))
    (and (subset-relation [(difference [?X0 ?X1]) ?X0])
         (forall (((difference [?X0 ?X1]) ?x))
             (not (?X1 ?x)))
         (forall ((?X0 ?x) (not (?X1 ?x)))
             ((difference [?X0 ?X1]) ?x))))
```

Here are a few of the many properties that relate these operations:

$$
\begin{aligned}
X_0 \cup X_1 &= X_1 \cup X_0 \\
(X_0 \cup X_1) \cup X_2 &= X_0 \cup (X_1 \cup X_2) \\
X_0 \cap X_1 &= X_1 \cap X_0 \\
(X_0 \cap X_1) \cap X_2 &= X_0 \cap (X_1 \cup X_2) \\
(X_0 \setminus X_1) \cup X_1 &= X_0 \cup X_1 \\
(X_0 \setminus X_1) \cap X_1 &= \emptyset
\end{aligned}
$$

$$
\begin{array}{rl}
\text{if} & X \subseteq Y, \\
\text{then} & X \cup Y = Y,\ X \cap Y = X \text{ and } X \setminus Y = \emptyset \\
\text{hence,} & X \cup X = X,\ X \cup \emptyset = X, \\
& X \cap X = X,\ X \cap \emptyset = \emptyset, \\
& X \setminus \emptyset = X,\ X \setminus X = \emptyset
\end{array}
$$

```
(forall ((set ?X0) (set ?X1))
    (and (= (binary-union [?X0 ?X1]) (binary-union [?X1 ?X0]))
         (= (binary-intersection [?X0 ?X1]) (binary-intersection [?X1 ?X0]))
         (= (binary-union [(difference [?X0 ?X1]) ?X1]) ?X1)
         (= (binary-intersection [(difference [?X0 ?X1]) ?X1]) zero)))

(forall ((set ?X0) (set ?X1) (set ?X2))
    (and (= (binary-union [(binary-union [?X0 ?X1]) ?X2])
            (binary-union [?X0 (binary-union [?X1 ?X2])]))
         (= (binary-intersection [(binary-intersection [?X0 ?X1]) ?X2])
            (binary-intersection [?X0 (binary-intersection [?X1 ?X2])]))))

(forall ((set ?X0) (set ?X1))
    (=> (subset-relation [?X0 ?X1])
        (and (= (binary-union [?X0 ?X1]) ?X1)
             (= (binary-intersection [?X0 ?X1]) ?X0)
             (= (difference [?X0 ?X1]) empty))))

(forall ((set ?X))
    (and (= (binary-union [?X ?X]) ?X) (= (binary-union [?X empty]) ?X)
         (= (binary-intersection [?X ?X]) ?X) (= (binary-intersection [?X empty]) empty)
         (= (difference [?X empty]) ?X) (= (difference [?X ?X]) empty)))
```

Instances. Here are some basic sets: $\mathbf{0} = \mathsf{zero} = \emptyset$, the initial empty set; $\mathbf{1} = \mathsf{one}$, the terminal set with one element; $\mathbf{2} = \mathsf{two} = \mathbf{1} + \mathbf{1}$ and $\mathbf{3} = \mathsf{three} = \mathbf{1} + \mathbf{1} + \mathbf{1}$. zero and one have several synonyms. two and three are often used for indexing. Nothing is in zero. The canonical object in one is denoted $0 \in \mathsf{one}$. The canonical object in two, but not one, is denoted $1 \in \mathsf{two} \backslash \mathsf{one}$. The canonical object in three, but not two, is denoted $2 \in \mathsf{three} \backslash \mathsf{two}$. These three canonical objects are distinct: $0 \neq 1$, $1 \neq 2$ and $0 \neq 2$. $\mathbf{0}$ has the universal property that for any set X there is only one function $\mathbf{0} \to X$, and $\mathbf{1}$ has the universal property that for any set X there is only one function $X \to \mathbf{1}$. There is a *constant zero*

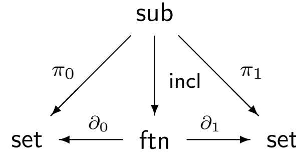
function $\Delta_{\mathsf{zero}} : \mathsf{set} \stackrel{!_{\mathsf{set}}}{\to} \mathbf{1} \stackrel{\mathsf{zero}}{\to} \mathsf{set}$, which maps any meta set X to the meta set $\mathsf{zero} = 0$. There is a *constant one* function $\Delta_{\mathsf{one}} : \mathsf{set} \stackrel{!_{\mathsf{set}}}{\to} \mathbf{1} \stackrel{\mathsf{one}}{\to} \mathsf{set}$, which maps any meta set X to the meta set $\mathsf{one} = 1$. For any type set X, there is a *counique* type function $\hat{!}_X : \emptyset \to X$ and a *unique* type function $!_X : X \to 1$. These are the unique functions between their respective sources and targets.

```
(iff:thing iff:0) (iff:thing iff:1) (iff:thing iff:2)
(not (= iff:0 iff:1)) (not (= iff:1 iff:2)) (not (= iff:0 iff:2))

(set zero) (set initial) (= initial zero) (set empty) (= empty initial)
(forall ((zero ?x)) (not (zero ?x)))

(set one) (set terminal) (= terminal one) (one 0)
(forall ((one ?x)) (= ?x 0))

(set two) (two 0) (two 1)
(forall ((two ?x)) (or (= ?x 0) (= ?x 1)))

(set three) (three 0) (three 1) (three 2)
(forall ((three ?x)) (or (= ?x 0) (= ?x 1) (= ?x 2)))

(iff:function constant-zero)
(= (iff:source constant-zero) set)
(= (iff:target constant-zero) set)
(forall ((set ?X))
    (= (constant-zero ?X) zero))

(iff:function constant-one)
(= (iff:source constant-one) set)
(= (iff:target constant-one) set)
(forall ((set ?X))
    (= (constant-one ?X) one))

(iff:function counique)
(= (iff:source counique) set)
(= (iff:target counique) type.ftn:function)
(forall ((set ?X))
    (and (= (type.ftn:source (counique ?X)) zero)
         (= (type.ftn:target (counique ?X)) ?X)
         (forall ((type.function ?f)
                 (= (type.ftn:source ?f) zero)
                 (= (type.ftn:target ?f) ?X))
             (= ?f (counique ?X)))))

(iff:function unique)
(= (iff:source unique) set)
(= (iff:target unique) type.ftn:function)
(forall ((set ?X))
    (and (= (type.ftn:source (unique ?X)) ?X)
         (= (type.ftn:target (unique ?X)) one)
         (forall ((type.function ?f)
                 (= (type.ftn:source ?f) ?X)
                 (= (type.ftn:target ?f) one))
             (= ?f (unique ?X)))))
```

The following inclusions (subset relationships) hold for instances: $\mathsf{zero} \subseteq X$ for any set X, and $\mathsf{one} \subset \mathsf{two} \subset \mathsf{three}$.

```
(forall ((set ?X)) (subset-relation [zero ?X]))
(subset-relation [one two]) (not (= one two))
(subset-relation [two three]) (not (= two three))
```

Subobject. There is a binary *subset* relation $\subseteq$ on type sets. In the subset relationship $Y \subseteq X$, all elements of the smaller type set Y are members of the larger type set X. A pair of type sets is subordinate when it satisfies the subset relation. Let $\mathsf{sub} = \mathsf{ext}(\subseteq)$ denote the set of type subordinate pairs. We name the components of a subset relationship. There are projections $\pi_0^{\subseteq} : \mathsf{sub} \rightarrow \mathsf{set}$ and $\pi_1^{\subseteq} : \mathsf{sub} \rightarrow \mathsf{set}$ to the *smaller* and *larger* components. For each subordinate pair of type sets $Y \subseteq X$, there is an *inclusion* injection $\iota_{Y,X} = \mathsf{incl}_{Y,X} : Y \hookrightarrow X : x \mapsto x$, whose source is the smaller type set and whose target is the larger type set. The subset relation on type sets is a partial order (reflexive, antisymmetric and transitive), since identities are inclusions and inclusions are closed under composition[3].




```
(iff:set subset-relation)
(forall ((subset-relation ?YX)) (type.dgm.pr.obj:set-pair ?YX))
(forall ((set ?Y) (set ?X))
    (<=> (subset-relation [?Y ?X])
         (forall ((?Y ?x)) (?X ?x))))

(iff:function smaller)
(= (iff:source smaller) subset-relation)
(= (iff:target smaller) set)
(forall ((set ?Y) (set ?X) (subset-relation [?Y ?X]))
    (= (smaller [?Y ?X]) ?Y))

(iff:function larger)
(= (iff:source larger) subset-relation)
(= (iff:target larger) set)
(forall ((set ?Y) (set ?X) (subset-relation [?Y ?X]))
    (= (larger [?Y ?X]) ?X))

(iff:function inclusion)
(= (iff:source inclusion) subset-relation)
(= (iff:target inclusion) type.ftn:function)
```

[3]We might be interested in extending the subset relation to a subobject relation. If we say that Y is a subobject of X, we mean that there is an injection $Y \hookrightarrow X$. There could be more than one of these. However, there is at least one and we could choose and name a canonical one of these. With this assumption, there would be a function $\mathsf{inj} : \mathsf{sub} \rightarrow \mathsf{ftn}$ such that $\mathsf{inj}_{Y,X} : Y \hookrightarrow X$ is an injection for any subordinate pair (Y, X). Furthermore, we could assume that the choosen injection is the inclusion when $Y \subseteq X$. This would allow us to conservatively extend the delimitation, restriction and abridgment relations. The main problem is that the chosen injections do not strictly obey transitivity; that is, for subordinate pairs (Z, Y) and (Y, X), we would have an isomorphism $\mathsf{inj}_{Z,Y} \cdot \mathsf{inj}_{Y,X} \cong \mathsf{inj}_{Z,X}$, but not necessarily an identity. And we need an identity, to show transitivity for delimitation, restriction and abridgment.

```
(forall ((set ?Y) (set ?X) (subset-relation [?Y ?X]))
    (and (= (type.ftn:source (inclusion [?Y ?X])) ?Y)
         (= (type.ftn:target (inclusion [?Y ?X])) ?X)
         (forall ((?Y ?y))
             (= ((inclusion [?Y ?X]) ?y) ?y))))

(forall ((set ?X))
    (subset-relation [?X ?X]))
(forall ((set ?Y) (set ?X))
    (=> (and (subset-relation [?Y ?X]) (subset-relation [?X ?Y]))
        (= ?Y ?X)))
(forall ((set ?Z) (set ?Y) (set ?X))
    (=> (and (subset-relation [?Z ?Y]) (subset-relation [?Y ?X]))
        (subset-relation [?Z ?X])))
```

The subset order can be defined in terms of Boolean operations.

$$\begin{aligned} Y \subseteq X \ \text{ iff } \ & Y \cap X = Y \\ \text{iff } \ & Y \cup X = X \\ \text{iff } \ & Y \setminus X = \emptyset \end{aligned}$$

```
(forall ((set ?Y) (set ?X))
    (<=> (subset-relation [?Y ?X])
         (= (binary-intersection [?Y ?X]) ?Y)))

(forall ((set ?Y) (set ?X))
    (<=> (subset-relation [?Y ?X])
         (= (binary-union [?Y ?X]) ?X)))

(forall ((set ?Y) (set ?X))
    (<=> (subset-relation [?Y ?X])
         (= (difference [?Y ?X]) empty)))
```

For every subordinate pair $Y \subseteq X$, there is a type *predicate* $\mathsf{pred}_{Y,X} : Y \hookrightarrow X$, whose genus is X, whose differentia is Y and whose function is the inclusion $\mathsf{incl}_{Y,X} : Y \hookrightarrow X$.

```
(iff:function predicate)
(= (iff:source predicate) subordinate)
(= (iff:target predicate) predicate)
(forall ((subordinate ?YX))
    (and (= (type.pred:genus (predicate ?YX)) (smaller ?YX))
         (= (type.pred:differentia (predicate ?YX)) (larger ?YX))
         (= (type.pred:function (predicate ?YX)) (inclusion ?YX))))
```

Any subordinate pair is identical to the subordinate pair of its predicate.

```
(forall ((subset-relation ?YX))
    (= ?YX (type.pred:subordinate (predicate ?YX))))
```

For any type set X, there is a *power* type set $\wp X$ consisting of the set of all subsets of X

$$\wp X \ = \ \{Y \in \mathsf{set} \mid Y \subseteq X\}.$$

The IFF power function $\wp : \mathsf{set} \rightarrow \mathsf{set}$ is the (implicit) 10-fiber of the subset relation. It is an injection, since $\wp X_1 = \wp X_2$ implies $X_1 = \cup(\wp X_1) = \cup(\wp X_2) =$

X_2 for any two sets $X_1, X_2 \in \mathsf{set}$. For convenience, here we name the product of the pairing of the power function. For any type set X, a *power pair* is a pair of X-subsets $(Y_0, Y_1) \in \wp X^2 = \wp X \times \wp X$.

$$\begin{aligned} \wp X^0 &= \mathsf{one} \\ \wp X^1 &= \wp X \\ \wp X^2 &= \wp X \times \wp X \end{aligned}$$

```
(iff:function power)
(= (iff:source power) set)
(= (iff:target power) set)
(forall ((set ?X) ((power ?X) ?Y)) (set ?Y))
(forall ((set ?X) (set ?Y))
    (<=> ((power ?X) ?Y)
         (subset-relation [?Y ?X])))

(iff:function power-pair)
(= (iff:source power-pair) set)
(= (iff:target power-pair) set)
(forall ((set ?X))
    (= (power-pair ?X) (type.lim.pwr2.obj:power (power ?X))))
```

Let X be any type set.

- The subset order on sets restricts to the power $\wp X$. There is a binary *leq* relationship $\subseteq_X$ between pairs of X-subsets in $\wp X$. One X-subset $Y \subseteq X$ is less than or equal to another X-subset $Z \subseteq X$, denoted $Y \subseteq_X Z$, when Y is a subset of Z: $Y \subseteq_X Z$ iff $Y \subseteq Z$. This local inclusion relation is a partial order (reflexive, antisymmetric and transitive).

- There is a *bottom* X-subset $\ulcorner \bot_X \urcorner : \Delta_{\mathsf{one}}(X) = 1 \to \wp X$. The bottom X-subset $\bot$ is the X-subset $\emptyset \subseteq X$ whose inclusion is the counique function $!_X : \emptyset \hookrightarrow X$; that is, for all elements $x \in X$, not $\bot(x)$.

- There is a *top* X-subset $\ulcorner \top_X \urcorner : \Delta_{\mathsf{one}}(X) = 1 \to \wp X$. The top X-subset $\top$ is the X-subset $X \subseteq X$ whose inclusion is the identity $1_X : X \hookrightarrow X$; that is, for all elements $x \in X$, $\top(x)$.

- There is a *binary join* (*disjunction*) operation $\vee_X : \wp X \times \wp X \to \wp X$. For any two X-subset $Y \subseteq X$ and $Z \subseteq X$, the binary join is the X-subset $Y \vee Z = \{x \in X \mid x \in Y \text{ or } x \in Z\} \subseteq X$.

- There is a *binary meet* (*conjunction*) operation $\wedge_X : \wp X \times \wp X \to \wp X$. For any two X-subset $Y \subseteq X$ and $Z \subseteq X$, the binary meet is the X-subset $Y \wedge Z = \{x \in X \mid x \in Y \text{ and } x \in Z\} \subseteq X$.

- There is a *minus* operation $\backslash_X : \wp X \times \wp X \to \wp X$. For any two X-subsets $Y \subseteq X$ and $Z \subseteq X$, the minus is the X-subset $Y \backslash Z = \{x \in X \mid x \in Y \text{ or } x \notin Z\} \subseteq X$. Clearly, $Y \setminus Z = Y \wedge \neg Z$.

- There is a *complement* (*negation*) operation $\neg_X : \wp X \to \wp X$. For any X-subset $Y \subseteq X$, the complement is the X-subset $\neg Y = \{x \in X \mid x \notin Y\} \subseteq X$, the minus of the "universe" X with Y, $\neg Y = X \setminus Y$.

- There is a *implication* operation $\Rightarrow_X : \wp X \times \wp X \rightarrow \wp X$. For any two X-subsets $Y \subseteq X$ and $Z \subseteq X$, the implication is the X-subset $Y \Rightarrow Z = \{x \in X \mid x \in Y \text{ implies } x \in Z\} = \{x \in X \mid x \in Z \text{ or } x \notin Y\} \in X$. Clearly, $Y \Rightarrow Z = \neg(Y \setminus Z) = \neg(Y \wedge \neg Z) = \neg Y \vee Z$.

```
(iff:function leq)
(= (iff:source leq) set)
(= (iff:target leq) type.rel:endorelation)
(forall ((set ?X))
    (and (= (type.rel:set (leq ?X)) (power ?X))
         (forall (((power ?X) ?Y) ((power ?X) ?Z))
             (<=> ((leq ?X) ?Y ?Z)
                  (subset-relation [?Y ?Z])))))
(forall ((set ?X))
    (type.rel:partial-order (leq ?X)))
(forall ((set ?X) ((power ?X) ?Y) ((power ?X) ?Z))
    (<=> ((leq ?X) ?Y ?Z)
         (= ((binary-meet ?X) [?Y ?Z]) ?Y)))

(iff:function bottom)
(= (iff:source bottom) set)
(= (iff:target bottom) type.ftn:function)
(forall ((set ?X))
    (and (= (type.ftn:source (bottom ?X)) (constant-one ?X))
         (= (type.ftn:target (bottom ?X)) (power ?X))
         (= ((bottom ?X) iff:0) empty)))

(iff:function top)
(= (iff:source top) set)
(= (iff:target top) type.ftn:function)
(forall ((set ?X))
    (and (= (type.ftn:source (top ?X)) (constant-one ?X))
         (= (type.ftn:target (top ?X)) (power ?X))
         (= ((top ?X) iff:0) ?X)))

(iff:function binary-join)
(= (iff:source binary-join) set)
(= (iff:target binary-join) type.ftn:function)
(forall ((set ?X))
    (and (= (type.ftn:source (binary-join ?X)) (power-pair ?X))
         (= (type.ftn:target (binary-join ?X)) (power ?X))
         (forall (((power ?X) ?Y) ((power ?X) ?Z))
             (= ((binary-join ?X) [?Y ?Z]) (binary-union [?Y ?Z])))))

(iff:function binary-meet)
(= (iff:source binary-meet) set)
(= (iff:target binary-meet) type.ftn:function)
(forall ((set ?X))
    (and (= (type.ftn:source (binary-meet ?X)) (power-pair ?X))
         (= (type.ftn:target (binary-meet ?X)) (power ?X))
         (forall (((power ?X) ?Y) ((power ?X) ?Z))
             (= ((binary-meet ?X) [?Y ?Z]) (binary-intersection [?Y ?Z])))))

(iff:function minus)
(= (iff:source minus) set)
(= (iff:target minus) type.ftn:function)
(forall ((set ?X))
```

```
    (and (= (type.ftn:source (minus ?X)) (power-pair ?X))
         (= (type.ftn:target (minus ?X)) (power ?X))
         (forall (((power ?X) ?Y) ((power ?X) ?Z))
             (and (= ((minus ?X) [?Y ?Z]) (difference [?Y ?Z]))
                  (= ((minus ?X) [?Y ?Z])
                     ((binary-meet ?X) [?Y ((complement ?X) ?Z)]))))))

(iff:function complement)
(= (iff:source complement) set)
(= (iff:target complement) type.ftn:function)
(forall ((set ?X))
    (and (= (type.ftn:source (complement ?X)) (power ?X))
         (= (type.ftn:target (complement ?X)) (power ?X))
         (forall (((power ?X) ?Y))
             (and (= ((complement ?X) ?Y) (difference [?X ?Y]))
                  (= ((complement ?X) ?Y) ((minus ?X) [(top ?X) ?Y]))))))

(iff:function implication)
(= (iff:source implication) set)
(= (iff:target implication) type.ftn:function)
(forall ((set ?X))
    (and (= (type.ftn:source (implication ?X)) (power-pair ?X))
         (= (type.ftn:target (implication ?X)) (power ?X))
         (forall (((power ?X) ?Y) ((power ?X) ?Z))
             (and (= ((implication ?X) [?Y ?Z])
                     (difference [?X (difference [?Y ?Z])]))
                  (= ((implication ?X) [?Y ?Z])
                     ((complement ?X) ((difference ?X) [?Y ?Z])))
                  (= ((implication ?X) [?Y ?Z])
                     ((binary-join ?X) [((complement ?X) ?Y) ?Z]))))))
```

For any type set X, the *singleton* type function $\{\text{-}\}_X : X \rightarrow \wp X$ is defined as

$$\{x\}_X \; = \; \{x\}$$

for any element $x \in \wp X$. The singleton operation $\{\text{-}\}_X$ is an injection.

```
(iff:function singleton)
(= (iff:source singleton) set)
(= (iff:target singleton) type.ftn:function)
(forall ((set ?X))
    (and (= (type.ftn:source (singleton ?X)) ?X)
         (= (type.ftn:target (singleton ?X)) (power ?X))
         (forall ((?X ?x) (?X ?y))
             (<=> (((singleton ?X) ?x) ?y) (= ?y ?x)))
         (type.ftn:injection (singleton ?X))))
```

For any type set X, the (bounded) *union* (or *join*) type function $\cup_X : \wp\wp X \rightarrow \wp X$ is defined as

$$\cup_X(Z) \; = \; \{x \in X \mid \exists_{Y \in Z} \, x \in Y\}$$

for any family of subsets $Z \in \wp\wp X$. The union operation $\cup_X$ is a surjection, since for any $Y \in \wp X$ we have $\{Y\} \in \wp\wp X$ and $\cup_X(\{Y\}) = Y$; that is, $\exists_{\{\text{-}\}_X} \cdot \cup_X = 1_{\wp X} : \wp X \rightarrow \wp\wp X \rightarrow \wp X$.

```
(iff:function union) (iff:function join) (= join union)
(= (iff:source union) set)
(= (iff:target union) type.ftn:function)
(forall ((set ?X))
    (and (= (type.ftn:source (union ?X)) (power (power ?X)))
         (= (type.ftn:target (union ?X)) (power ?X))
         (forall (((power (power ?X)) ?Z) (?X ?x))
             (<=> (((union ?X) ?Z) ?x)
                  (exists ((?Z ?Y)) (?Y ?x))))
         (type.ftn:surjection (union ?X))))

(forall ((set ?X) ((power ?X) ?Y))
    (= (type.ftn:composition [(exists (singleton ?X)) (union ?X)])
       (type.ftn:identity (power ?X))))
```

The triple $\langle \wp, \cup, \{\text{-}\} \rangle$ forms a(n implicit) monad. This means that $\wp : \mathsf{Set} \rightarrow \mathsf{Set}$ is a(n implicit) functor, $\cup : \wp \circ \wp \Rightarrow \wp$ and $\{\text{-}\} : 1_{\mathsf{Set}} \Rightarrow \wp$ are (implicit) natural transformations, and these satisfy the associative law and two unit laws

$$\begin{aligned} \cup_{\wp(X)} \cdot \cup_X &= \wp(\cup_X) \cdot \cup_X \\ \wp(\{\text{-}\}_X) \cdot \cup_X &= 1_{\wp(X)} \\ \{\text{-}\}_{\wp(X)} \cdot \cup_X &= 1_{\wp(X)} \end{aligned}$$

for all sets $X \in \mathsf{set}$.

```
(forall ((set ?X))
    (and (= (type.ftn:composition [(union (power ?X)) (union ?X)])
            (type.ftn:composition [(type.ftn:power (union ?X)) (union ?X)]))
         (= (type.ftn:composition [(type.ftn:power (singleton ?X)) (union ?X)])))
            (type.ftn:identity (power ?X)))
         (= (type.ftn:composition [(singleton (power ?X)) (union ?X)])))
            (type.ftn:identity (power ?X)))))
```

For any type set X, the *intersection* (or *meet*) type function $\cap_X : \wp\wp X \rightarrow \wp X$ is defined as

$$\cap_X(Z) \;=\; \{x \in X \mid \forall_{Y \in Z}\, x \in Y\}$$

for any family of subsets $Z \in \wp\wp X$. The intersection operation $\cap_X$ is a surjection, since for any $Y \in \wp X$ we have $\{Y\} \in \wp\wp X$ and $\cap_X(\{Y\}) = Y$; that is, $\exists_{\{\text{-}\}_X} \cdot \cap_X = 1_{\wp X} : \wp X \rightarrow \wp\wp X \rightarrow \wp X$.

```
(iff:function intersection) (iff:function meet) (= meet intersection)
(= (iff:source intersection) set)
(= (iff:target intersection) type.ftn:function)
(forall ((set ?X))
    (and (= (type.ftn:source (intersection ?X)) (power (power ?X)))
         (= (type.ftn:target (intersection ?X)) (power ?X))
         (forall (((power (power ?X)) ?Z) (?X ?x))
             (<=> (((intersection ?X) ?Z) ?x)
                  (forall ((?Z ?Y)) (?Y ?x))))
         (type.ftn:surjection (intersection ?X))))

(forall ((set ?X) ((power ?X) ?Y))
    (= (type.ftn:composition [(exists (singleton ?X)) (intersection ?X)])
       (type.ftn:identity (power ?X))))
```

Let $f : X \rightarrow Y$ be any type function. There is an internal *exist*ential quantification, (direct) image or (*power*) type function $\exists_f : \wp X \rightarrow \wp Y$ defined by

$$y \in \exists_f A \;\equiv\; \left\{ \begin{array}{l} \text{“there exists something } x \text{ in } X \\ \text{that satisfies } x \in A \textit{ and} \\ \text{goes to } y \text{ under } f : f(x) = y\text{”} \end{array} \right.$$

or

$$\exists_f(A) \;=\; \{y \in Y \mid \exists_{x \in X}(x \in A \wedge f(x) = y)\}$$

for any source subset $A \subseteq X$. There is an internal *inverse image* type function

$$f^{-1} : \wp Y \rightarrow \wp X$$

defined by

$$\begin{array}{rcl} f^{-1}(B) & = & \{x \in X \mid \exists_{y \in B}(f(x) = y)\} \\ & = & \{x \in X \mid f(x) \in B\} \end{array}$$

for any target subset $B \subseteq Y$. There is an internal universal quantification (*forall*) type function $\forall_f : \wp X \rightarrow \wp Y$ defined by

$$y \in \forall_f A \;\equiv\; \left\{ \begin{array}{l} \text{“for all things } x \text{ in } X \\ \text{for which } f(x) = y, \\ x \in A \text{ holds”} \end{array} \right.$$

or

$$\forall_f(A) \;=\; \{y \in Y \mid \forall_{x \in X}(f(x) = y \Rightarrow A(x))\}$$

for any source subset $A \subseteq X$. The existential (inverse image, universal) function is monotonic. The existential quantification operation is functorial: the existential quantification of the identity $1_X : X \rightarrow X$ is the identity of the power, $\exists_{1_X} = 1_{\wp X} : \wp X \rightarrow \wp X$; and the existential quantification of the composition $f \cdot g : X \rightarrow Y \rightarrow Z$ is the composition of the existential quantifications, $\exists_{f \cdot g} = \exists_f \cdot \exists_g : \wp X \rightarrow \wp Z$. The inverse image (universal quantification) operation is also functorial. In the diagram

$$\wp X \;\; \begin{array}{c} \xrightarrow{\exists_f} \\ \xleftarrow[f^{-1}]{} \\ \xrightarrow[\forall_f]{} \end{array} \;\; \wp Y$$

we have the adjunctions $\exists_f \dashv f^{-1} \dashv \forall_f$; that is, the existential quantification is left adjoint to the inverse image and the inverse image is left adjoint to the universal quantification.

```
(iff:function exists) (iff:function power) (= power exists)
(= (iff:source exists) type.ftn:function)
(= (iff:target exists) type.ftn:function)
(forall ((type.ftn:function ?f))
    (and (= (type.ftn:source (exists ?f)) (power (type.ftn:source ?f)))
         (= (type.ftn:target (exists ?f)) (power (type.ftn:target ?f)))
         (forall (((power (type.ftn:source ?f)) ?A) ((type.ftn:target ?f) ?y))
```

```
            (<=> (((exists ?f) ?A) ?y)
                 (exists (((type.ftn:source ?f) ?x))
                     (and (?A ?x) (= (?f ?x) ?y)))))))

(iff:function inverse-image)
(= (iff:source inverse-image) type.ftn:function)
(= (iff:target inverse-image) type.ftn:function)
(forall ((type.ftn:function ?f))
    (and (= (type.ftn:source (inverse-image ?f)) (power (type.ftn:target ?f)))
         (= (type.ftn:target (inverse-image ?f)) (power (type.ftn:source ?f)))
         (forall (((power (type.ftn:target ?f)) ?B) ((type.ftn:source ?f) ?x))
             (<=> (((inverse-image ?f) ?B) ?x)
                  (?B (?f ?x))))))

(iff:function forall)
(= (iff:source forall) type.ftn:function)
(= (iff:target forall) type.ftn:function)
(forall ((type.ftn:function ?f))
    (and (= (type.ftn:source (forall ?f)) (power (type.ftn:source ?f)))
         (= (type.ftn:target (forall ?f)) (power (type.ftn:target ?f)))
         (forall (((power (type.ftn:source ?f)) ?A) ((type.ftn:target ?f) ?y))
             (<=> (((forall ?f) ?A) ?y)
                  (forall (((type.ftn:source ?f) ?x))
                      (=> (= (?f ?x) ?y) (?A ?x)))))))

(forall ((type.ftn:function ?f))
    (and (forall (((power (type.ftn:source ?f)) ?X0)
                  ((power (type.ftn:source ?f)) ?X1)
                 (subset-relation [?X0 ?X1]))
             (subset-relation [((exists ?f) ?X0) ((exists ?f) ?X1)]))
         (forall (((power (type.ftn:target ?f)) ?Y0)
                  ((power (type.ftn:target ?f)) ?Y1)
                 (subset-relation [?Y0 ?Y1]))
             (subset-relation [((inverse-image ?f) ?Y0) ((inverse-image ?f) ?Y1)]))
         (forall (((power (type.ftn:source ?f)) ?X0)
                  ((power (type.ftn:source ?f)) ?X1)
                 (subset-relation [?X0 ?X1]))
             (subset-relation [((forall ?f) ?X0) ((forall ?f) ?X1)]))))

(forall ((type.ftn:function ?f))
    (and (forall (((power (type.ftn:source ?f)) ?X)
                  ((power (type.ftn:target ?f)) ?Y))
             (<=> (subset-relation [((exists ?f) ?X) ?Y])
                  (subset-relation [?X ((inverse-image ?f) ?Y)])))
         (forall (((power (type.ftn:source ?f)) ?X)
                  ((power (type.ftn:target ?f)) ?Y))
             (<=> (subset-relation [((inverse-image ?f) ?Y) ?X])
                  (subset-relation [?Y ((forall ?f) ?X)])))))

(forall ((set ?X))
    (= (exists (type.ftn:identity ?X)) (type.ftn:identity (power ?X))))

(forall ((type.ftn:function ?f) (type.ftn:function ?g) (composable-pair [?f ?g]))
    (= (exists (type.ftn:composition [?f ?g]))
       (type.ftn:composition [(exists ?f) (exists ?g)])))
```

Our two standard references for the IFF are the books: *Sets for Mathematics* (2003) by F. William Lawvere and Robert Rosebrugh [1] and *Categories for the Working Mathematician* (1971) by Saunders Mac Lane [2].

References

[1] F. William Lawvere and Robert Rosebrugh. *Sets for Mathematics*. Cambridge University Press, 2003.

[2] Saunders Mac Lane. *Categories for the Working Mathematician*. Springer-Verlag, 1971.